

A UNIFIED FLOW APPROACH TO SMOOTH, EVEN L_p -MINKOWSKI PROBLEMS

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ABSTRACT. We study long-time existence and asymptotic behaviour for a class of anisotropic, expanding curvature flows. For this we adapt new curvature estimates, which were developed by Guan, Ren and Wang to treat some stationary prescribed curvature problems. As an application we give a unified flow approach to the existence of smooth, even L_p -Minkowski problems in \mathbb{R}^{n+1} for $p > -n - 1$.

1. INTRODUCTION

Consider a smooth, closed, strictly convex hypersurface M_0 in Euclidean space \mathbb{R}^{n+1} , $n \geq 2$, given by a smooth embedding $F_0 : M \rightarrow \mathbb{R}^{n+1}$. Suppose the origin is in the interior of the region enclosed by M_0 . We study the long-time behaviour of a family of hypersurfaces $\{M_t\}$ given by smooth maps $F : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$ satisfying the initial value problem

$$(1.1) \quad \partial_t F(x, t) = \varphi(\nu(x, t)) \frac{(F(x, t) \cdot \nu(x, t))^{2-p}}{\mathcal{K}(x, t)} \nu(x, t), \quad F(\cdot, 0) = F_0(\cdot).$$

Here $\mathcal{K}(\cdot, t)$ and $\nu(\cdot, t)$ are the Gauss curvature and the outer unit normal vector of $M_t = F(M, t)$ and φ is a positive, smooth function on \mathbb{S}^n . Furthermore, T is the maximal time for which the solution exists.

For $p = 2$, $\varphi \equiv 1$, flow (1.1) was studied by Schnürer [60] in \mathbb{R}^3 and by Gerhardt [29] in higher dimensions. Both works rely on the reflection principle of Chow and Gulliver [21] and McCoy [47]. Their result is as follows: the volume-normalized flow evolves any M_0 in the C^∞ -topology to an origin-centered ball. For $p > 2$ and $\varphi \equiv 1$ it follows from Chow-Gulliver [21, Theorem 3.1] (see also Tsai [61, Example 1]) that (1.1) evolves M_0 , after rescaling to fixed volume, in the C^1 -topology to an origin-centered ball. We refer the reader to the paper [35] regarding a rather comprehensive list of previous works on this curvature flow. In particular, in either case $\varphi \neq 1$ or $\varphi \equiv 1$, $-n - 1 < p < 2$, we are not aware of any result in the literature on the asymptotic behaviour of the flow. The following theorem was proved in [35] regarding the case $p = -n - 1$, $\varphi \equiv 1$ (in this case the flow belongs to a family of centro-affine normal flows introduced by Stancu in [57]).

Let us set $\tilde{K}_t = (V(B)/V(K_t))^{1/(n+1)} K_t$, where K_t denotes the convex body enclosed by M_t and $V(K_t)$ is the $(n + 1)$ -dimensional Lebesgue measure of K_t .

Theorem ([35]). *Let $n \geq 2$, $p = -n - 1$, $\varphi \equiv 1$ and suppose K_0 has its Santaló point at the origin, i.e.,*

$$\int_{\mathbb{S}^n} \frac{u}{h_{K_0}(u)^{n+2}} d\sigma(u) = 0.$$

Then there exists a unique solution $\{M_t\}$ of flow (1.1), such that \tilde{M}_t converges in C^∞ to an origin-centered ellipsoid.

Here h_{K_0} is the support function of K_0 . A closed, convex hypersurface M_0 can be described in terms of its support function $h_{K_0} : \mathbb{S}^n \rightarrow \mathbb{R}$ defined by

$$h_{K_0}(u) = \sup\{u \cdot x : x \in M_0\}.$$

If M_0 is smooth and strictly convex, then $h_{K_0}(u) = u \cdot F_0(\nu^{-1}(u))$.

From the evolution equation of $F(\cdot, t)$ it follows that

$$h(\cdot, t) := h_{K_t}(\cdot) : \mathbb{S}^n \times [0, T) \rightarrow \mathbb{R}$$

evolves by

$$(1.2) \quad \partial_t h(u, t) = \varphi(u)(h^{2-p} S_n)(u, t),$$

where $S_n(u, t) = 1/\mathcal{K}(\nu^{-1}(u, t), t)$. A homothetic self-similar solution of this flow satisfies

$$(1.3) \quad h^{1-p} \det(\bar{\nabla}^2 h + \text{Id } h) = \frac{c}{\varphi},$$

for some positive constant c . Here $\bar{\nabla}$ is the covariant derivative on \mathbb{S}^n . Note that $S_n = \det(\bar{\nabla}^2 h + \text{Id } h)$. In particular, self-similar solutions to the flow (1.1) are solutions of the L_p -Minkowski problem, (1.4), which we shall introduce now.

The Minkowski problem deals with existence, uniqueness, regularity, and stability of closed convex hypersurfaces whose Gauss curvature (as a function of the outer normals) is preassigned. Major contributions to this problem were made by Minkowski [48, 49], Aleksandrov [2–4], Fenchel and Jessen [25], Lewy [40, 41], Nirenberg [50], Calabi [15], Pogorelov [51, 52], Cheng and Yau [18], Caffarelli, Nirenberg, and Spruck [16], and others. A generalization of the Minkowski problem known as the L_p -Minkowski problem was introduced by Lutwak [42, 43]. This generalization was studied in [42, 44]. Solutions to many cases of these generalized problems followed later in [1, 6, 11, 12, 14, 17, 20, 24, 26, 27, 31, 33, 38, 39, 45, 46, 54–56, 62, 65–67].

In the smooth category, the L_p -Minkowski problem asks, given a smooth, strictly positive function $\varphi : \mathbb{S}^n \rightarrow \mathbb{R}$, does there exist a smooth, closed, strictly convex hypersurface $M_0 \subset \mathbb{R}^{n+1}$ such that,

$$(1.4) \quad \frac{1}{\mathcal{K}(x)} = \frac{h^{p-1}(\nu(x))}{\varphi(\nu(x))}$$

where $x \in M_0$, h denotes the support function, \mathcal{K} the Gauss curvature and ν the Gauss map $M_0 \rightarrow \mathbb{S}^n$. The *even* L_p -Minkowski problem requires in addition, that φ is an even function. The case $p = 1$ is the original Minkowski problem.

The existence and regularity of solutions to the L_p -Minkowski problem are rather comprehensively discussed in [20] for $p > n - 1$. Our study on (1.1) provides an alternative variational treatment (based on curvature flow) of the even L_p -Minkowski problem. For $p = 1$, Chou-Wang [19] treated the classical L_1 -Minkowski problem in the smooth category by a logarithmic Gauss curvature flow. For $n = 1$, and $1 \neq p > -3$, the existence of solutions to the L_p -Minkowski problems follows from Andrews' results [9] on the asymptotic behaviour of a family of contracting and expanding flows of curves. Also, in higher dimensions, the existence of solutions to the L_p -Minkowski problems follows from [11] when $-n - 1 < p \leq -n + 1$ (a short proof of this is also given in [36]) or when φ is even (e.q., $\varphi(u) = \varphi(-u)$) and $-n + 1 < p < 1$. See also [5, 10, 30, 63, 64].

We now list the main results of the paper extending the previous mentioned results.

Theorem 1 (Even L_p -Minkowski problem). *Let $-n - 1 < p < \infty$ and φ be a positive, smooth even function on \mathbb{S}^n i.e., $\varphi(u) = \varphi(-u)$. Suppose K_0 is origin-symmetric. There exists a unique origin-symmetric solution $\{M_t\}$ of (1.1) such that $\{\tilde{M}_t\}$ converges for a subsequence of times in C^1 to a smooth, origin-symmetric, strictly convex solution of (1.3). Also, when $p \leq n + 1$ the convergence is in C^∞ , and if $p \geq 1$ the convergence holds for the full sequence.*

If $-n - 1 < p \leq -n$, we can extend the result of the previous theorem by dropping the assumption that φ is even. Therefore for this range of p , as in [11], we can establish the existence of solutions to the smooth L_p -Minkowski problem.

Theorem 2. *Let $-n - 1 < p \leq -n$ and K_0 satisfy*

$$\int_{\mathbb{S}^n} \frac{u}{\varphi(u)h_{K_0}(u)^{1-p}} d\sigma(u) = 0.$$

There exists a unique solution $\{M_t\}$ of flow (1.1) such that $\{\tilde{M}_t\}$ converges for a subsequence of times in C^∞ to a positive, smooth, strictly convex solution of (1.3).

For $\varphi \equiv 1$ we prove the following theorem.

Theorem 3. *Let $1 \neq p > -n - 1$, $\varphi \equiv 1$ and K_0 satisfy*

$$\int_{\mathbb{S}^n} \frac{u}{h_{K_0}(u)^{1-p}} d\sigma(u) = 0.$$

Then there exists a unique solution $\{M_t\}$ of (1.1) such that $\{\tilde{M}_t\}$ converges for a subsequence of times in C^1 to a positive, smooth, strictly convex solution of (1.3). In addition, for $1 \neq p \leq n + 1$ the convergence holds in C^∞ , and when $p > 1$ the full sequence converges to the unit ball.

Remark 4. Two remarks are in order:

- (1) Given any convex body K_0 , there exists a vector \vec{v} such that $K_0 + \vec{v}$ has the origin in its interior and it satisfies the assumption of the second theorem.
- (2) In the third theorem, in some other cases than $p \geq 1$, it is known that the limiting shape is the unit ball. See, for example, [10, 13]. Moreover, as it was mentioned before, in view of the reflection principle, for $p \geq 2$ the assumption $\int_{\mathbb{S}^n} \frac{u}{h_{K_0}(u)^{1-p}} d\sigma(u) = 0$ is not needed. Therefore, our main contribution for the case $2 < p \leq n + 1$ is the C^∞ convergence which is established in Section 4.2

The main difficulty in proving convergence of the normalized solutions is in obtaining long-time existence. The issue arises from the time-dependent anisotropic factor (the support function). We believe in such generality, (1.1) serves as the first example where a time-dependent anisotropic factor is allowed. To prove long-time existence, we first obtain bounds on the Gauss curvature in Section 3.1 using the well known standard technique of Tso [58] to obtain upper bounds. We obtain lower bounds by applying the same technique to the evolution of the polar body as in [36]. Controlling the principal curvatures requires estimates of higher derivatives of the speed which is generally quite difficult due to the non-linearity of the flow. In Section 3.2 we obtain these crucial estimates by adapting the remarkable C^2 estimates

of Guan-Ren-Wang for the prescribed curvature problem see [32, (4.2)]. Long time existence then follows readily by standard arguments. Once it is proved that solutions to the flow exist until they expand to infinity uniformly in all directions, the method of [35, Section 8] applies and yields convergence of the volume-normalized solutions in C^1 to self-similar solutions provided $p \neq 1$. Further work is required to establish convergence of normalized solutions if $p = 1$, and to prove convergence in C^∞ for $p \leq n + 1$; this is accomplished in Section 4; see also Remark 10.

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2. BASIC EVOLUTION EQUATIONS

Let $g = \{g_{ij}\}$, and $W = \{w_{ij}\}$ denote, in order, the induced metric and the second fundamental form of M . At every point in the hypersurface M choose a local orthonormal frame $\{e_1, \dots, e_n\}$.

We use the following standard notation

$$\begin{aligned} w_i^j &= g^{mj} w_{im}, \\ (w^2)_i^j &= g^{mj} g^{rs} w_{ir} w_{sm}, \\ |W|^2 &= g^{ij} g^{kl} w_{ik} w_{lj} = w_{ij} w^{ij}. \end{aligned}$$

Here, $\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$.

We use semicolons to denote covariant derivatives. The following geometric formulas are well-known:

$$\begin{aligned} \nu_{;i} &= w_i^k e_k, \\ \nu_{;ij} &= g^{kl} w_{ij;l} e_k - w_i^l w_{lj} \nu, \\ h_{;i} &= w_i^k (F \cdot e_k), \\ h_{;ij} &= w_{ij} - h w_i^l w_{lj} + F \cdot \nabla w_{ij}. \end{aligned}$$

Note that in above we considered the support function as a function on the boundary of the hypersurface; that is, at the point $x \in M$ we have

$$h(x) = F(x) \cdot \nu(x).$$

For convenience, let $\psi(x) = h^{2-p}(x) \varphi(\nu(x))$. The following evolution can be deduced in a standard manner; see for example [28].

Lemma 5. *The following evolution equations hold:*

$$\partial_t \nu = -\nabla \left(\frac{\psi}{\mathcal{K}} \right),$$

$$\begin{aligned}
\partial_t w_i^j &= - \left(\frac{\psi}{\mathcal{K}} \right)_{;ik} g^{kj} - \left(\frac{\psi}{\mathcal{K}} \right) w_i^k w_k^j \\
&= \psi \frac{\mathcal{K}^{kl}}{\mathcal{K}^2} w_{i;kl}^j + \psi \frac{\mathcal{K}^{kl}}{\mathcal{K}^2} w_{kr} w_l^r w_i^j - (n+1) \frac{\psi}{\mathcal{K}} w_i^k w_k^j \\
&\quad + \psi \frac{\mathcal{K}^{kl,rs}}{\mathcal{K}^2} g^{jm} w_{kl;i} w_{rs;m} - \frac{2\psi}{\mathcal{K}^3} g^{jm} \mathcal{K}_{;i} \mathcal{K}_{;m} \\
&\quad + \frac{1}{\mathcal{K}^2} g^{jk} \mathcal{K}_{;k} \psi_{;i} + \frac{1}{\mathcal{K}^2} g^{jk} \psi_{;k} \mathcal{K}_{;i} - \frac{1}{\mathcal{K}} g^{jk} \psi_{;ik}, \\
\partial_t h &= \psi \frac{\mathcal{K}^{ij}}{\mathcal{K}^2} h_{;ij} + \psi h \frac{\mathcal{K}^{ij}}{\mathcal{K}^2} w_i^l w_{lj} - (n-1) \frac{\psi}{\mathcal{K}} - \frac{1}{\mathcal{K}} F \cdot \nabla \psi.
\end{aligned}$$

3. LONG-TIME EXISTENCE

3.1. Lower and upper bounds on Gauss curvature. The proofs of the following two lemmas are similar to the proofs of [36, Lemmas 4.1, 4.2]. For completeness, we give the proofs here. In this section we use $\bar{\nabla}$ to denote covariant derivatives on the sphere with respect to the standard metric.

The matrix of the radii of the curvature of a smooth, closed, strictly convex hypersurface is denoted by $\mathbf{r} = [\mathbf{r}_{ij}]$ and the entries of \mathbf{r} are considered as functions on the unit sphere. They can be expressed in terms of the support function as $\mathbf{r}_{ij} := \bar{\nabla}_{ij}^2 h + \bar{g}_{ij} h$, where $[\bar{g}_{ij}]$ is the standard metric on \mathbb{S}^n . Additionally, we recall that $S_n = \det[\mathbf{r}_{ij}] / \det[\bar{g}_{ij}]$.

Lemma 6. *Let $\{M_t\}$ be a solution of (1.1) on $[0, t_1]$. If $c_2 \leq h_{K_t} \leq c_1$ on $[0, t_1]$, then $\mathcal{K} \leq c_4$ on $[0, t_1]$. Here c_4 depends on $K_0, c_1, c_2, p, \varphi$ and t_1 .*

Proof. We apply the maximum principle to the following auxiliary function defined on the unit sphere

$$\Theta = \frac{\psi S_n}{2c_1 - h} = \frac{\partial_t h}{2c_1 - h}.$$

At any minimum of Θ we have

$$0 = \bar{\nabla}_i \Theta = \bar{\nabla}_i \left(\frac{\psi S_n}{2c_1 - h} \right) \quad \text{and} \quad \bar{\nabla}_{ij}^2 \Theta \geq 0.$$

Therefore,

$$\frac{\bar{\nabla}_i(\psi S_n)}{2c_1 - h} = - \frac{\psi S_n \bar{\nabla}_i h}{(2c_1 - h)^2}$$

and

$$(3.1) \quad \bar{\nabla}_{ij}^2(\psi S_n) + \bar{g}_{ij} \psi S_n \geq \frac{-\psi S_n \mathbf{r}_{ij} + 2c_1 \psi S_n \bar{g}_{ij}}{2c_1 - h}.$$

Differentiating Θ with respect to time yields

$$\partial_t \Theta = \frac{\psi S_n^{ij}}{2c_1 - h} (\bar{\nabla}_{ij}^2(\psi S_n) + \bar{g}_{ij} \psi S_n) + \frac{\psi^2 S_n^2}{(2c_1 - h)^2} (1 + (2-p)h^{-1}(2c_1 - h)),$$

where S_n^{ij} is the derivative of S_n with respect to the entry \mathbf{r}_{ij} . By applying inequality (3.1) to the preceding identity we deduce

$$(3.2) \quad \partial_t \Theta \geq \Theta^2 (1 - n + 2c_1 \mathcal{H}) - c \Theta^2,$$

where

$$\mathcal{H} = S_n^{-1} S_n^{ij} \bar{g}_{ij}.$$

Therefore

$$\frac{\varphi^{\frac{h^{2-p}}{\mathcal{K}}}}{2c_1 - h}(t, u) \geq \frac{1}{ct + 1/\min_{u \in \mathbb{S}^n} \frac{\varphi^{\frac{h^{2-p}}{\mathcal{K}}}}{2c_1 - h}(0, u)} \geq \frac{1}{ct_1 + 1/\min_{u \in \mathbb{S}^n} \frac{\varphi^{\frac{h^{2-p}}{\mathcal{K}}}}{2c_1 - h}(0, u)}.$$

□

Lemma 7. *Let $\{M_t\}$ be a solution of (1.1) on $[0, t_1]$. If $c_1 \leq h_{K_t} \leq c_2$ on $[0, t_1]$, then $\mathcal{K} \geq \frac{1}{a+bt - \frac{n}{n+1}}$ on $(0, t_1]$, where a and b depend only on c_1, c_2, p, φ . In particular, $\mathcal{K} \geq c_3$ on $[0, t_1]$ for a positive number c_3 that depends on $K_0, c_1, c_2, p, \varphi$ and is independent of t_1 .*

Proof. Suppose K_t^* is the polar body¹ of K_t with respect to the origin. We furnish quantities associated with polar bodies with $*$. The polar bodies evolve by

$$\partial_t h^* = -\psi^* S_n^{*-1}, \quad h^*(\cdot, t) = h_{K_t^*}(\cdot),$$

where

$$\psi^* = \frac{(h^{*2} + |\bar{\nabla} h^*|^2)^{\frac{n+1+p}{2}}}{h^{*n+1}} \varphi \left(\frac{h^* u + \bar{\nabla} h^*}{\sqrt{h^{*2} + |\bar{\nabla} h^*|^2}} \right);$$

see Lemma 11 for the proof. In addition, we have $c'_1 = 1/c_2 \leq h^* \leq 1/c_1 = c'_2$. We will show that the function

$$\Theta = \frac{\psi^* S_n^{*-1}}{h^* - c'_1/2}$$

remains bounded. At any maximal point of Θ :

$$0 = \bar{\nabla}_i \Theta = \bar{\nabla}_i \left(\frac{\psi^* S_n^{*-1}}{h^* - c'_1/2} \right) \quad \text{and} \quad \bar{\nabla}_{ij}^2 \Theta \leq 0.$$

Hence, we obtain

$$(3.3) \quad \frac{\bar{\nabla}_i(\psi^* S_n^{*-1})}{h^* - c'_1/2} = \frac{\psi^* S_n^{*-1} \bar{\nabla}_i h^*}{(h^* - c'_1/2)^2},$$

and consequently,

$$(3.4) \quad \bar{\nabla}_{ij}^2(\psi^* S_n^{*-1}) + \bar{g}_{ij} \psi^* S_n^{*-1} \leq \frac{\psi^* S_n^{*-1} \mathbf{r}_{ij}^* - c'_1/2 \psi^* S_n^{*-1} \bar{g}_{ij}}{h^* - c'_1/2}.$$

Differentiating Θ with respect to time yields

$$\begin{aligned} \partial_t \Theta &= \frac{\psi^* S_n^{*-2}}{h^* - c'_1/2} S_n^{*ij} (\bar{\nabla}_{ij}^2(\psi^* S_n^{*-1}) + \bar{g}_{ij} \psi^* S_n^{*-1}) \\ &\quad + \frac{S_n^{*-1}}{h^* - c'_1/2} \partial_t \psi^* + \Theta^2. \end{aligned}$$

On the other hand, since

$$|\partial_t h^*| = \psi^* S_n^{*-1}, \quad \|\bar{\nabla} \partial_t h^*\| = \|\bar{\nabla}(\psi^* S_n^{*-1})\| = \frac{\psi^* S_n^{*-1} \|\bar{\nabla} h^*\|}{h^* - c'_1/2}, \quad \|\bar{\nabla} h^*\| \leq c'_2,$$

¹The polar body of a convex body K with the origin of \mathbb{R}^{n+1} in its interior is the convex body defined by

$$K^* = \{x \in \mathbb{R}^{n+1} | x \cdot y \leq 1 \text{ for all } y \in K\}.$$

where for the second equation we used (3.3), we have

$$\frac{S_n^{*-1}}{h^* - c'_1/2} \partial_t \psi^* \leq c(n, p, c_1, c_2, \varphi) \Theta^2.$$

Employing this last inequality and inequality (3.4) we infer that, at any point where the maximum of Θ is reached, we have

$$(3.5) \quad \partial_t \Theta \leq \Theta^2 \left(c' - \frac{c'_1}{2} \mathcal{H}^* \right).$$

Moreover,

$$\begin{aligned} \mathcal{H}^* &\geq n \left(\frac{h^* - c'_1/2}{\psi^* S_n^{*-1}} \right)^{-\frac{1}{n}} \left(\frac{\psi^*}{h^* - c'_1/2} \right)^{-\frac{1}{n}} \\ &\geq n \Theta^{\frac{1}{n}} \left(\frac{c''}{c'_1 - c'_1/2} \right)^{-\frac{1}{n}}. \end{aligned}$$

Therefore, we can rewrite the inequality (3.5) as follows

$$\partial_t \Theta \leq \Theta^2 \left(c - c' \Theta^{\frac{1}{n}} \right),$$

for positive constants c and c' depending only on p, c_1, c_2, φ . Hence,

$$(3.6) \quad \Theta \leq c + c' t^{-\frac{n}{n+1}}$$

for some positive constants depending only on p, c_1, c_2, φ .² The inequality (3.6) implies that

$$(3.8) \quad S_n^{*-1} \leq a' + b' t^{-\frac{n}{n+1}}$$

for some a' and b' depending only on p, c_1, c_2, φ . Now we can use the argument given in [37, Lemma 2.3] to obtain the desired lower bound: For every $u \in \mathbb{S}^n$, there exists a unique $u^* \in \mathbb{S}^n$ such that

$$(S_n h^{n+2})(u) (S_n^* h^{*n+2})(u^*) = 1,$$

2

Claim. Suppose f is a positive smooth function of t on $[0, t_1]$ that satisfies

$$(3.7) \quad \frac{d}{dt} f \leq c_0 + c_1 f + c_2 f^2 - c_3 f^{2+p},$$

where c_3, p are positive. There exists constant $c, c' > 0$ independent of the solution and depending only on c_0, c_1, c_2, c_3, p , such that $f \leq c + c' t^{-1/(p+1)}$ on $(0, t_1]$,

Proof. Note that there exists $x_0 > 0$ such that $c_0 + c_1 x + c_2 x^2 - c_3 x^{2+p} < -c_3/2 x^{2+p}$ for $x > x_0$. If $f(0) \leq x_0$, then f may increase forward in time, but when f reaches x_0 , then f must start decreasing (since the right-hand side of (3.7) becomes negative). Thus we may assume, without loss of generality, that $f(0) > x_0$. Therefore, $f > x_0$ on a maximal time interval $[0, t_0)$. On $[0, t_0)$ we can solve

$$\frac{d}{dt} f \leq -c_3/2 f^{2+p}$$

to obtain

$$f \leq (c_3(p+1)/2t)^{-1/(p+1)}.$$

At t_0 we have $c_0 + c_1 f + c_2 f^2 - c_3 f^{1+p} = -c_3/2 f^{2+p}$ and $f = x_0$; therefore the right-hand side of (3.7) is still negative. So $f \leq f(t_0)$ on $[t_0, t_1]$. In conclusion,

$$f \leq \max\{(c_3(p+1)/2t)^{-1/(p+1)}, x_0 = f(t_0)\} \leq c + c' t^{-1/(1+p)},$$

where c, c' do not depend on solutions. \square

see [34]. In view of this identity and (3.8) we conclude that on $(0, t_1]$ we have

$$\mathcal{K} \geq \frac{1}{a + bt^{-\frac{n}{n+1}}}$$

for some a and b depending only on p, c_1, c_2, φ . The lower bound for \mathcal{K} on $[0, \delta]$ for a small enough $\delta > 0$ follows from the short-time existence of the flow. The lower bound for \mathcal{K} on $[\delta, t_1]$ follows from the inequality $\mathcal{K} \geq \frac{1}{a+b\delta^{-\frac{n}{n+1}}}$. \square

3.2. Upper and lower bounds on principal curvatures. To obtain upper and lower bounds on the principal curvatures, we will consider the auxiliary function used by Guan-Ren-Wang for a prescribed curvature problem; see [32, (4.2)].

Lemma 8. *Let $\{M_t\}$ be a solution of (1.1) on $[0, t_1]$. If $c_1 \leq h_{K_t} \leq c_2$ on $[0, t_1]$, then $c_5 \leq \kappa_i \leq c_6$ on $[0, t_1]$, where c_5 and c_6 depend on $K_0, c_1, c_2, p, \varphi$ and t_1 .*

Proof. In view of Lemmas 6 and 7, it suffices to show that $\|W\|$ remains bounded on $[0, t_1]$. Consider the auxiliary function

$$\Theta = \frac{1}{2} \log(\|W\|^2) - \alpha \log h.$$

Assume without loss of generality that $c_1 > 1$, for otherwise we replace h by $2h/c_1$, which does not effect the evolution equation of Θ . Using the parabolic maximum principle we show that for some α large enough $\Theta(\cdot, t)$ is always negative on $[0, t_1]$. If the conclusion of the theorem is false, we may choose (x_0, t_0) with $t_0 > 0$ and such that $\Theta(x_0, t_0) = 0$, $\Theta(x, t_0) \leq 0$, and $\Theta(x, t) < 0$ for $t < t_0$. Then,

$$\begin{aligned} 0 &\leq \dot{\Theta} - \psi \frac{\mathcal{K}^{kl}}{\mathcal{K}^2} \Theta_{;kl} \\ &= -\frac{\psi}{\|W\|^2} \frac{\mathcal{K}^{kl}}{\mathcal{K}^2} w_{;i,k}^j w_{j;l}^i + \frac{2\psi}{\|W\|^4} \frac{\mathcal{K}^{kl}}{\mathcal{K}^2} w_i^j w_r^s w_{j;k}^i w_{s;l}^r \\ &\quad + \psi \frac{\mathcal{K}^{kl}}{\mathcal{K}^2} w_{kr} w_l^r - (n+1) \psi \frac{(w^2)_i^j w_j^i}{\mathcal{K} \|W\|^2} \\ &\quad + \frac{\psi w_j^i}{\|W\|^2} \left(\frac{\mathcal{K}^{kl,rs}}{\mathcal{K}^2} w_{kl;i} g^{jp} j w_{rs;p} - 2 \frac{g^{jp} \mathcal{K}_{;i} \mathcal{K}_{;p}}{\mathcal{K}^3} \right) \\ &\quad + \left(\frac{2}{\mathcal{K}^2} g^{jp} \psi_{;i} \mathcal{K}_{;j} - \frac{1}{\mathcal{K}} g^{jp} \psi_{;ip} \right) \frac{w_j^i}{\|W\|^2} \\ &\quad + (n-1) \frac{\alpha \psi}{h \mathcal{K}} + \frac{\alpha}{h \mathcal{K}} (F \cdot \nabla \psi) - \frac{\alpha \psi}{h^2} \frac{\mathcal{K}^{kl}}{\mathcal{K}^2} h_{;k} h_{;l} - \alpha \psi \frac{\mathcal{K}^{kl}}{\mathcal{K}^2} w_{kr} w_l^r. \end{aligned}$$

Working in an orthonormal frame at (x_0, t_0) , with respect to which we may write

$$\mathcal{K}^{kl,rs} w_{kl;i} w_{rs;i} = \mathcal{K}^{kk,ll} w_{kk;i} w_{ll;i} - \mathcal{K}^{kk,ll} w_{kl;i}^2,$$

due to the relation

$$\begin{aligned} \mathcal{K}^{kl,rs} w_{kl;i} w_{rs;j} w^{ij} &= \sum_i w_{ii} \left(\sum_{p,q} \frac{\partial^2 \mathcal{K}}{\partial \kappa_p \partial \kappa_q} w_{pp;i} w_{qq;i} \right. \\ &\quad \left. + \sum_{p \neq q} \frac{\frac{\partial \mathcal{K}}{\partial \kappa_p} - \frac{\partial \mathcal{K}}{\partial \kappa_q}}{\kappa_p - \kappa_q} w_{pq;i}^2 \right), \end{aligned} \tag{3.9}$$

see for example [28, Lemma 2.1.14]. We obtain after multiplication by \mathcal{K}^2 that

$$\begin{aligned}
0 \leq & -\frac{\psi}{\|W\|^2} \mathcal{K}^{ii} \sum_l w_{ll;i}^2 - \frac{\psi}{\|W\|^2} \mathcal{K}^{ii} \sum_{p \neq q} w_{pq;i}^2 + \frac{2\psi}{\|W\|^4} \mathcal{K}^{ii} \left(\sum_j w_{jj} w_{jj;i} \right)^2 \\
& + \psi \mathcal{K}^{ii} w_{ii}^2 - (n+1) \psi \mathcal{K} \sum_i \frac{w_{ii}^3}{\|W\|^2} \\
& + \frac{\psi}{\|W\|^2} \sum_i w_{ii} \left(\mathcal{K}^{pp,qq} w_{pp;i} w_{qq;i} - \mathcal{K}^{pp,qq} w_{pq;i}^2 - 2 \frac{(\mathcal{K}_{;i})^2}{\mathcal{K}} \right) \\
& + \sum_i (2\psi_{;i} \mathcal{K}_{;i} - \mathcal{K} \psi_{;ii}) \frac{w_{ii}}{\|W\|^2} \\
& + (n-1) \frac{\alpha \psi \mathcal{K}}{h} + \frac{\alpha \mathcal{K}}{h} (F \cdot \nabla \psi) - \frac{\alpha \psi}{h^2} \mathcal{K}^{kl} h_{;k} h_{;l} - \alpha \psi \mathcal{K}^{ii} w_{ii}^2.
\end{aligned}$$

From the vanishing of the first variation, at (x_0, t_0) we have

$$(3.10) \quad 0 = \Theta_{;k} = \sum_i \frac{w_{ii} w_{ii;k}}{\|W\|^2} - \alpha \frac{h_{;k}}{h},$$

We may assume at x_0 that $w_1^1 = \max\{w_{ii} : 1 \leq i \leq n\}$. Therefore,

$$(3.11) \quad \Theta(x_0, t_0) = 0 \Rightarrow \frac{c_1^\alpha}{\sqrt{n}} \leq w_{11} \leq c_2^\alpha.$$

On the other hand, since ψ is bounded above and below in view of the hypotheses of the lemma, we obtain

$$\begin{aligned}
\psi_{;i} \leq C_0 w_{ii} & \Rightarrow 2\psi_{;i} \mathcal{K}_{;i} \leq \frac{\varepsilon \psi}{c_4} (\mathcal{K}_{;i})^2 + \frac{c_4 C_0^2}{\psi \varepsilon} w_{ii}^2 \\
(3.12) \quad & \leq \varepsilon \psi \frac{(\mathcal{K}_{;i})^2}{\mathcal{K}} + C(\varepsilon, K_0, \varphi, t_1) \psi w_{ii}^2,
\end{aligned}$$

where c_4 (depending on t_1) is from Lemma 6, and

$$(3.13) \quad \psi_{;ii} \geq -C - C w_{ii} - C w_{ii}^2 + \sum_k w_{ii;k} d_\nu \psi(\partial_k).$$

Using (3.10) in (3.13) we obtain

$$\begin{aligned}
& -\frac{\mathcal{K}}{\|W\|^2} \sum_i w_{ii} \psi_{;ii} \\
(3.14) \quad & \leq \frac{\mathcal{K}}{\|W\|^2} \sum_i w_{ii} (C + C w_{ii} + C w_{ii}^2 - \sum_k w_{ii;k} d_\nu \psi(\partial_k)) \\
& \leq \frac{\mathcal{K}}{\|W\|^2} \sum_i w_{ii} (C + C w_{ii} + C w_{ii}^2) - \frac{\alpha \mathcal{K}}{h} \sum_k h_{;k} d_\nu \psi(\partial_k) \\
& = \frac{\mathcal{K}}{\|W\|^2} \sum_i w_{ii} (C + C w_{ii} + C w_{ii}^2) - \frac{\alpha \mathcal{K}}{h} \sum_i w_{ii} (\partial_i \cdot F) d_\nu \psi(\partial_i) \\
& \leq \frac{\psi}{\|W\|^2} \sum_i w_{ii} (C + C w_{ii}^2) - \frac{\alpha \mathcal{K}}{h} \sum_i w_{ii} (\partial_i \cdot F) d_\nu \psi(\partial_i).
\end{aligned}$$

For the last inequality, we used that \mathcal{K} is bounded above and ψ is bounded below (so the constant C depends on K_0, φ, t_1).

Combining (3.12) and (3.14) implies that

$$\begin{aligned}
(3.15) \quad 0 &\leq -\frac{\psi}{\|W\|^2} \mathcal{K}^{ii} \sum_l w_{ll;i}^2 - \frac{\psi}{\|W\|^2} \mathcal{K}^{ii} \sum_{p \neq q} w_{pq;i}^2 \\
&\quad + \frac{2\psi}{\|W\|^4} \mathcal{K}^{ii} \left(\sum_j w_{jj} w_{jj;i} \right)^2 + \psi \mathcal{K}^{ii} w_{ii}^2 - (n+1) \psi \mathcal{K} \sum_i \frac{w_{ii}^3}{\|W\|^2} \\
&\quad + \frac{\psi}{\|W\|^2} \sum_l w_{ll} \left(\mathcal{K}^{pp,qq} w_{pp;l} w_{qq;l} - \mathcal{K}^{pp,qq} w_{pq;l}^2 - (2-\varepsilon) \frac{(\mathcal{K}_{;l})^2}{\mathcal{K}} \right) \\
&\quad + \frac{\psi}{\|W\|^2} \sum_i w_{ii} (C + C w_{ii}^2) - \frac{\alpha \mathcal{K}}{h} \sum_i w_{ii} (\partial_i \cdot F) d_\nu \psi(\partial_i) \\
&\quad + (n-1) \frac{\alpha \psi \mathcal{K}}{h} + \frac{\alpha \mathcal{K}}{h} \sum_s (\partial_s \cdot F) d_F \psi(\partial_s) + \frac{\alpha \mathcal{K}}{h} \sum_i w_{ii} (\partial_i \cdot F) d_\nu \psi(\partial_i) \\
&\quad - \frac{\alpha \psi}{h^2} \mathcal{K}^{ii} w_{ii}^2 (\partial_i \cdot F)^2 - \alpha \psi \mathcal{K}^{ii} w_{ii}^2 \\
&\leq \frac{\psi}{\|W\|^2} \left(\sum_l w_{ll} (C + C w_{ll}^2) - n \mathcal{K} \sum_l w_{ll}^3 + \mathcal{K}^{ii} w_{ii}^2 \|W\|^2 \right) \\
&\quad + \alpha \psi \left(\frac{n \mathcal{K}}{h} - \mathcal{K}^{ii} w_{ii}^2 - \frac{\mathcal{K}^{ii} w_{ii}^2 (\partial_i \cdot F)^2}{h^2} + \frac{\mathcal{K}}{h \psi} \sum_s (\partial_s \cdot F) d_F \psi(\partial_s) \right) \\
&\quad - \psi \sum_i (A_i + B_i + C_i + D_i - E_i) - \frac{\alpha \psi \mathcal{K}}{h} - \psi \mathcal{K} \sum_i \frac{w_{ii}^3}{\|W\|^2},
\end{aligned}$$

where C depends on $\varepsilon, K_0, \varphi, t_1$, and

$$\begin{aligned}
A_i &= \frac{2-\varepsilon}{\|W\|^2 \mathcal{K}} w_{ii} (\mathcal{K}_{;i})^2 - \frac{w_{ii}}{\|W\|^2} \sum_{p,q} \mathcal{K}^{pp,qq} w_{pp;i} w_{qq;i}, \\
B_i &= \frac{2}{\|W\|^2} \sum_j w_{jj} \mathcal{K}^{jj,ii} w_{jj;i}^2, \quad C_i = \frac{2}{\|W\|^2} \sum_{j \neq i} \mathcal{K}^{jj} w_{jj;i}^2, \\
D_i &= \frac{1}{\|W\|^2} \mathcal{K}^{ii} \sum_j w_{jj;i}^2, \quad E_i = \frac{2}{\|W\|^4} \mathcal{K}^{ii} \left(\sum_j w_{jj} w_{jj;i} \right)^2.
\end{aligned}$$

The terms B_i and C_i deserve some explanation. C_i comes from the second term in (3.15), which reads

$$-\frac{\psi}{\|W\|^2} \sum_i \mathcal{K}^{ii} \sum_{p \neq q} w_{pq;i}^2 \leq -\frac{\psi}{\|W\|^2} \sum_{p \neq q} \mathcal{K}^{pp} w_{pq;p}^2 - \frac{\psi}{\|W\|^2} \sum_{p \neq q} \mathcal{K}^{qq} w_{pq;q}^2,$$

which is exactly C_i due to the Codazzi equation.

The third line of (3.15) arises from (3.9). Since the second term in the bracket of (3.9) is negative and the hypersurface is convex, we can proceed in the same way as we derived C_i and just throw away all indices i which are neither p or q . This gives term B_i . The first term in the big bracket goes into A_i .

In Corollary 14 of the appendix we will present an adaption of the method developed in [32] to deal with the curvature derivative terms A_i, B_i, C_i, D_i, E_i . There

we prove that we obtain the following alternative: There exist positive numbers $\delta_2, \dots, \delta_n$, which only depend on the dimension, such that either

$$w_{ii} > \delta_i w_{11} \quad \forall 2 \leq i \leq n$$

or

$$A_i + B_i + C_i + D_i - E_i \geq 0 \quad \forall 1 \leq i \leq n.$$

By taking α large in (3.11), in the first case we get a contradiction to the bound on the Gauss curvature. In the second case, using also $\mathcal{K}^{ii} w_{ii}^2 = \mathcal{K} \sum_i w_{ii}$, (3.15) yields

$$\begin{aligned} 0 \leq & \frac{\psi}{\|W\|^2} \left(\sum_l w_{ll} (C + C w_{ll}^2) - n \mathcal{K} \sum_l w_{ll}^3 \right) - (\alpha - 1) \mathcal{K} \psi \sum_i w_{ii} \\ & + \alpha \psi \left((n-1) \frac{\mathcal{K}}{h} - \frac{\mathcal{K}}{h^2} \sum_i w_{ii} (\partial_i \cdot F)^2 + \frac{\mathcal{K}}{h \psi} \sum_l (\partial_l \cdot F) d_F \psi (\partial_l) \right). \end{aligned}$$

Consequently we obtain

$$0 \leq \frac{C(\varepsilon, K_0, \varphi, t_1) w_{11}^3}{\|W\|^2} - (\alpha - 1) \mathcal{K} \psi w_{11} + C(K_0, \varphi, t_1) \alpha,$$

where we discarded $-(\alpha - 1) \mathcal{K} \psi \sum_{i \neq 1} \kappa_i \leq 0$ and used the bounds on h, ψ and \mathcal{K} to bound κ_1 in terms of κ_1^3 .

Now take α such that $(\alpha - 1) \mathcal{K} \psi \geq C(\varepsilon, K_0, \varphi, t_1) + 1$. Therefore, in view of (3.11)

$$\begin{aligned} (3.16) \quad 0 & \leq \frac{C(\varepsilon, K_0, \varphi, t_1) w_{11}^3}{\|W\|^2} - (\alpha - 1) \mathcal{K} \psi w_{11} + C(K_0, \varphi, t_1) \alpha \\ & \leq C(\varepsilon, K_0, \varphi, t_1) \left(\frac{w_{11}^2}{\|W\|^2} - 1 \right) w_{11} - w_{11} + C(K_0, \varphi, t_1) \alpha \\ & \leq -\frac{c_1^\alpha}{\sqrt{n}} + C(K_0, \varphi, t_1) \alpha. \end{aligned}$$

Taking α large enough yields a contradiction. \square

Proposition 9. *The solution to (1.1) satisfies $\lim_{t \rightarrow T} \max h_{K_t} = \infty$.*

Proof. First, let $p \geq n + 1$. In this case, by comparing with suitable outer balls, the flow exists on $[0, \infty)$. For $p > n + 1$, consider an origin centered ball B_r , such that $K_0 \supseteq B_r$. Then $K_t \supseteq B_{r(t)}$, where

$$r(t) = ((\min h_{K_0})^{p-n-1} + t(p-n-1) \min \varphi)^{\frac{1}{p-n-1}}$$

and $B_{r(t)}$ expands to infinity as t approaches ∞ . For $p = n + 1$, $K_t \supseteq B_{r(t)}$ with $r(t) = e^{t \min \varphi} \min h_{K_0}$ and $B_{r(t)}$ expands to infinity as t approaches ∞ .

Second, if $p < n + 1$, then the flow exists only on a finite time interval. If $\max h_{K_t} < \infty$, then by Lemmas 6, 7 and 8, the evolution equation (1.1) is uniformly parabolic on $[0, T)$. Thus, the result of Krylov and Safonov [59] and standard parabolic theory allow us to extend the solution smoothly past time T , contradicting its maximality. \square

4. CONVERGENCE OF NORMALIZED SOLUTIONS

4.1. **Convergence in C^1** , $1 \neq p > -n - 1$. By the proof of [35, Corollary 7.5], there exist r, R such that

$$(4.1) \quad 0 < r \leq h_{\tilde{K}_t} \leq R < \infty.$$

Therefore, a subsequence of $\{\tilde{M}_{t_k}\}$ converges in the Hausdorff distance to a limiting shape \tilde{M}_∞ with the origin in its interior. The argument of [35, Section 8.1] implies

$$\varphi h_{\tilde{K}_\infty}^{1-p} f_{\tilde{M}_\infty} = c,$$

where $f_{\tilde{M}_\infty}$ is the positive continuous curvature function of \tilde{M}_∞ and c is some positive constant. By [35, Fact 8.1], \tilde{K}_∞ is smooth and strictly convex. The C^1 -convergence follows, which is purely geometric and does not depend on the evolution equation, from [8, Lemma 13].

Remark 10. Section 4.1 completes the discussion on the existence of solutions to the smooth, even L_p -Minkowski problems in \mathbb{R}^{n+1} for $1 \neq p > -n - 1$. The next section discusses the C^∞ convergence when $1 \neq p \leq n + 1$, and also when $p = 1$ and solutions are origin-symmetric.

4.2. **Convergence in C^∞** . By [35, Lemma 9.2], there is a uniform upper bound on the Gauss curvature of the normalized solution when $p \leq n + 1$. In the following, we first obtain a uniform lower bound on the Gauss curvature of the normalized solution \tilde{K}_t .

Let $h : \mathbb{S}^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$ be a solution of equation (1.2). Then for each $\lambda > 0$, \bar{h} defined by

$$\begin{aligned} \bar{h} : \mathbb{S}^n \times \left[0, T/\lambda^{\frac{1+n-p}{n+1}}\right) &\rightarrow \mathbb{R}^{n+1} \\ \bar{h}(u, t) &= \lambda^{\frac{1}{n+1}} h\left(u, \lambda^{\frac{1+n-p}{n+1}} t\right) \end{aligned}$$

is also a solution of evolution equation (1.2) but with the initial data $\lambda^{\frac{1}{n+1}} h(\cdot, 0)$.

For each *fixed* time $t \in [0, T)$, define \bar{h} a solution of (1.2) as follows

$$\bar{h}(u, \tau) = \left(\frac{V(B)}{V(K_t)}\right)^{\frac{1}{n+1}} h\left(u, t + \left(\frac{V(B)}{V(K_t)}\right)^{\frac{1+n-p}{n+1}} \tau\right).$$

Note that $\bar{h}(\cdot, 0)$ is the support function of $(V(B)/V(K_t))^{\frac{1}{n+1}} K_t$; therefore,

$$r \leq \bar{h}(u, 0) \leq R.$$

Write \bar{K}_τ for the convex body associated with $\bar{h}(\cdot, \tau)$ and let B_c denote the ball of radius c centered at the origin. Since B_R encloses \bar{K}_0 , the comparison principle implies that B_{2R} will enclose \bar{K}_τ for $\tau \in [0, \delta]$, where δ depends only on p, R, ψ . By the first statement of Lemma 7 applied to \bar{h} , there is a uniform lower bound (depending only on r, R, p, φ) on the Gauss curvature of $\bar{K}_{\frac{\delta}{2}}$.

On the other hand, the volume of $\bar{K}_{\frac{\delta}{2}}$ is bounded above by $V(B_{2R})$; therefore,

$$\frac{V(B)}{V(B_{2R})} \leq c_t := \frac{V(K_t)}{V\left(K_{t + \left(\frac{V(B)}{V(K_t)}\right)^{\frac{1+n-p}{n+1}} \frac{\delta}{2}}}\right)} \leq 1$$

for all $t \in [0, T)$. Consequently,

$$\left(\frac{V(B)}{V\left(K_{t+\left(\frac{V(B)}{V(K_t)}\right)^{\frac{1+n-p}{n+1}}\frac{\delta}{2}}}\right)^{\frac{1}{n+1}}} h\left(u, t + \left(\frac{V(B)}{V(K_t)}\right)^{\frac{1+n-p}{n+1}}\frac{\delta}{2}\right) = c_t^{\frac{1}{n+1}} \bar{h}\left(\cdot, \frac{\delta}{2}\right)$$

has Gauss curvature bounded below for all $t \in [0, T)$.

Now we show that for every $\tilde{t} \in \left[(V(B)/V(K_0))^{\frac{1+n-p}{n+1}}\frac{\delta}{2}, T\right)$, we can find $t \in [0, T)$ such that

$$\tilde{t} = t + \left(\frac{V(B)}{V(K_t)}\right)^{\frac{1+n-p}{n+1}}\frac{\delta}{2}.$$

Define $f(t) = t + \left(\frac{V(B)}{V(K_t)}\right)^{\frac{1+n-p}{n+1}}\frac{\delta}{2} - \tilde{t}$ on $[0, T)$. f is continuous, and

$$\begin{cases} f(T) = T - \tilde{t} > 0, & p < n+1 \\ f(\infty) = \infty, & p = n+1 \\ f(0) \leq 0 & p \leq n+1. \end{cases}$$

The claim follows.

Next we obtain uniform lower and upper bounds on the principal curvatures of the normalized solution.

Consider the convex bodies $\tilde{K}_\tau := \left(\frac{V(B)}{V(K_t)}\right)^{\frac{1}{n+1}} K_t$, where

$$\tau(t) := \int_0^t \left(\frac{V(K_s)}{V(B)}\right)^{\frac{1+n-p}{n+1}} ds, \quad \textcolor{red}{3}$$

Let us furnish all geometric quantities associated with \tilde{K}_τ by an over-tilde. The evolution equation of \tilde{h}_τ is given by

$$\partial_\tau \tilde{h}_\tau = \varphi \tilde{h}^{2-p} \tilde{S}_n - \frac{\int_{\mathbb{S}^n} \varphi \tilde{h}^{2-p} \tilde{S}_n^2 d\sigma}{(n+1)V(B)} \tilde{h}.$$

Since $\frac{\int_{\mathbb{S}^n} \varphi \tilde{h}^{2-p} \tilde{S}_n^2 d\sigma}{(n+1)V(B)}$ is uniformly bounded above, applying the maximum principle to $\Theta = \frac{1}{2} \log(\|\tilde{W}\|^2) - \alpha \log \tilde{h}$, and arguing as in the proof of Lemma 8, we see that $\|\tilde{W}\|$ has a uniform upper bound. This in turn, in view of our lower and upper bounds on the Gauss curvature of \tilde{K}_τ , implies that we have uniform lower and upper bounds on the principal curvatures of \tilde{K}_τ . Higher order regularity estimates and convergence in C^∞ for a subsequence of $\{\tilde{K}_\tau\}$ follow from Krylov-Safonov [59], standard parabolic theory and the Arzelà-Ascoli theorem. The convergence for the

³Suppose $p < n+1$. For each $t \in [0, T)$ by the comparison principle we have

$$\frac{(\max h_{K_t})^{p-n-1}}{(n+1-p) \max \varphi} \leq T-t \leq \frac{(\min h_{K_t})^{p-n-1}}{(n+1-p) \min \varphi}.$$

Therefore, since $\frac{\max h_{K_t}}{\min h_{K_t}} \leq \frac{R}{r}$ (see (4.1)), we get

$$c_1(T-t)^{\frac{1}{p-n-1}} \leq \min h_{K_t} \leq \left(\frac{V(K_t)}{V(B)}\right)^{\frac{1}{n+1}} \leq \max h_{K_t} \leq c_2(T-t)^{\frac{1}{p-n-1}}.$$

Thus $\lim_{t \rightarrow T} \tau(t) = \infty$.

full sequence when $p \geq 1$ follows from the uniqueness of the self-similar solutions to (1.3); see [20, 42].

5. APPENDIX

Evolution of polar bodies. Let K be a smooth, strictly convex body with the origin in its interior. Suppose ∂K is parameterised by the radial function r . The metric $[g_{ij}]_{1 \leq i, j \leq n}$, unit normal ν , support function h , and the second fundamental form $[w_{ij}]_{1 \leq i, j \leq n}$ of ∂K can be written in terms of r and its partial derivatives as follows:

$$\begin{aligned} \mathbf{a}: g_{ij} &= r^2 \bar{g}_{ij} + \bar{\nabla}_i r \bar{\nabla}_j r, \\ \mathbf{b}: \nu &= \frac{rz - \bar{\nabla} r}{\sqrt{r^2 + \|\bar{\nabla} r\|^2}}, \\ \mathbf{c}: h &= \frac{r^2}{\sqrt{r^2 + \|\bar{\nabla} r\|^2}}, \\ \mathbf{d}: w_{ij} &= \frac{-r \bar{\nabla}_{ij}^2 r + 2 \bar{\nabla}_i r \bar{\nabla}_j r + r^2 \bar{g}_{ij}}{\sqrt{r^2 + \|\bar{\nabla} r\|^2}}. \end{aligned}$$

Since $\frac{1}{r}$ is the support function of K^* , we can calculate the entries of $[\mathbf{r}_{ij}^*]_{1 \leq i, j \leq n}$:

$$\mathbf{r}_{ij}^* = \bar{\nabla}_{ij}^2 \frac{1}{r} + \frac{1}{r} \bar{g}_{ij} = \frac{-r \bar{\nabla}_{ij}^2 r + 2 \bar{\nabla}_i r \bar{\nabla}_j r + r^2 \bar{g}_{ij}}{r^3}.$$

Thus, using (d) we get

$$\mathbf{r}_{ij}^* = \frac{\sqrt{r^2 + \|\bar{\nabla} r\|^2}}{r^3} w_{ij}.$$

Lemma 11. *As ∂K_t evolve by (1.2), their polars ∂K_t^* evolve as follows:*

$$\partial_t h^* = -\varphi \left(\frac{h^* u + \bar{\nabla} h^*}{\sqrt{h^{*2} + \|\bar{\nabla} h^*\|^2}} \right) \frac{(h^{*2} + \|\bar{\nabla} h^*\|^2)^{\frac{n+1+p}{2}}}{h^{*n+1} S_n^*}, \quad h^*(\cdot, t) := h_{K_t^*}(\cdot).$$

Proof. To obtain the evolution equation of $h_{K_t^*}$, we first need to parameterise M_t over the unit sphere

$$F = r(z(\cdot, t), t) z(\cdot, t) : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1},$$

where $r(z(\cdot, t), t)$ is the radial function of M_t in the direction $z(\cdot, t)$. Note that

$$\partial_t r = \varphi \frac{h^{2-p}}{\mathcal{K}} \frac{\sqrt{r^2 + \|\bar{\nabla} r\|^2}}{r},$$

and

$$\mathcal{K} = \frac{\det w_{ij}}{\det g_{ij}}, \quad \frac{1}{S_n^*} = \frac{\det \bar{g}_{ij}}{\det \mathbf{r}_{ij}^*}, \quad \frac{\det \bar{g}_{ij}}{\det g_{ij}} = \frac{1}{r^{2n-2} (r^2 + \|\bar{\nabla} r\|^2)},$$

$$h = \frac{1}{\sqrt{h^{*2} + \|\bar{\nabla} h^*\|^2}}.$$

Now we calculate

$$\begin{aligned}
\partial_t h^* &= \partial_t \frac{1}{r} \\
&= -\frac{h^{2-p}}{\mathcal{K}} \frac{\sqrt{r^2 + \|\bar{\nabla} r\|^2}}{r^3} \varphi \circ \nu \\
&= -h^{2-p} \frac{\sqrt{r^2 + \|\bar{\nabla} r\|^2}}{r^3} \frac{\det g_{ij}}{\det w_{ij}} \varphi \circ \nu \\
&= -h^{2-p} \frac{\sqrt{r^2 + \|\bar{\nabla} r\|^2}}{r^3} \frac{\det \bar{g}_{ij}}{\det \mathbf{r}_{ij}^*} \frac{\det g_{ij}}{\det \bar{g}_{ij}} \frac{\det \mathbf{r}_{ij}^*}{\det w_{ij}} \varphi \circ \nu \\
&= -\left(\frac{\sqrt{r^2 + \|\bar{\nabla} r\|^2}}{r^3} \right)^{n+1} \frac{r^{2n-2} (r^2 + \|\bar{\nabla} r\|^2)}{(h^{*2} + \|\bar{\nabla} h^*\|^2)^{\frac{2-p}{2}}} \frac{\varphi \circ \nu}{S_n^*}.
\end{aligned}$$

Replacing r by $1/h^*$ and taking into account **(b)** finishes the proof. \square

Estimates for curvature derivatives. For convenience we present some of the main ideas, how one can prove the alternative in Lemma 8 about balancing the curvature derivatives. This method was used in [32] for a similar stationary prescribed curvature equation. Recall that

$$\begin{aligned}
A_i &= \frac{2-\varepsilon}{\|W\|^2 \mathcal{K}} w_{ii} (\mathcal{K}_{;i})^2 - \frac{w_{ii}}{\|W\|^2} \sum_{p,q} \mathcal{K}^{pp,qq} w_{pp;i} w_{qq;i}, \\
B_i &= \frac{2}{\|W\|^2} \sum_j w_{jj} \mathcal{K}^{jj,ii} w_{jj;i}^2, \quad C_i = \frac{2}{\|W\|^2} \sum_{j \neq i} \mathcal{K}^{jj} w_{jj;i}^2, \\
D_i &= \frac{1}{\|W\|^2} \mathcal{K}^{ii} \sum_j w_{jj;i}^2, \quad E_i = \frac{2}{\|W\|^4} \mathcal{K}^{ii} \left(\sum_j w_{jj} w_{jj;i} \right)^2.
\end{aligned}$$

Note that the term A_i looks slightly different from the term A_i in [32, p. 1309], where the \mathcal{K} is not present in the denominator. We have to define A_i in the way we did, because due to the inverse nature of the curvature flow equation we obtain an extra good derivative term. This allows us to choose the constant in A_i as $2-\varepsilon$, whereas a large constant was required in [32] (denoted by K there). Fortunately the proofs of [32, Lemma 4.2, Lemma 4.3] also work for sufficiently small ε . The remaining terms B_i, C_i, D_i, E_i are all identical to those in [32].

In the following σ_k denotes the k -th elementary symmetric function of principal curvatures. We begin by recalling the following special case ($k = n$) of inequality (2.4) from [32, Lemma 2.2], which can be deduced easily by differentiating

$$\log G = \log \left(\frac{\sigma_n}{\sigma_l} \right)^{\frac{1}{n-l}}$$

twice, using the concavity of G and applying the Schwarz inequality. For any $\delta > 0$, $1 \leq i \leq n$ and $1 \leq l < n$ we have

$$\begin{aligned}
& -\mathcal{K}^{pp,qq} w_{pp;i} w_{qq;i} + \left(1 - \frac{1}{n-l} + \frac{1}{(n-l)\delta} \right) \frac{(\mathcal{K}_{;i})^2}{\mathcal{K}} \geq \\
& \left(1 + \frac{1-\delta}{n-l} \right) \frac{\mathcal{K}((\sigma_l)_{;i})^2}{\sigma_l^2} - \frac{\mathcal{K}}{\sigma_l} \sigma_l^{pp,qq} w_{pp;i} w_{qq;i}.
\end{aligned}$$

In particular, by taking $\delta = \frac{1}{2-\varepsilon}$, we have

$$(5.1) \quad (2-\varepsilon) \frac{(\mathcal{K}_{;i})^2}{\mathcal{K}} - \mathcal{K}^{pp,qq} w_{pp;i} w_{qq;i} \geq \left[1 + \frac{1-\varepsilon}{(n-1)(2-\varepsilon)} \right] \frac{\mathcal{K}((\sigma_l)_{;i})^2}{\sigma_l^2} - \frac{\mathcal{K} \sigma_l^{pp,qq} w_{pp;i} w_{qq;i}}{\sigma_l},$$

provided $(2-\varepsilon) \geq 1$, i.e. $0 < \varepsilon < 1$.

Lemma 12. *For each $i \neq 1$, if $\sqrt{3}\kappa_i \leq \kappa_1$, we have*

$$A_i + B_i + C_i + D_i - E_i \geq 0.$$

Proof. Note that from (5.1) with $l = 1$, it follows that $A_i \geq 0$ since $\sigma_1^{pp,qq} = 0$. The proof of that $B_i + C_i + D_i - E_i \geq 0$ can literally be taken from [32, Lemma 4.2], starting with [32, Equ. (4.10)]. \square

In the following proof we will write $\sigma_n = \mathcal{K}$ for a better comparability with [32, Lemma 4.3]. Also denote by $\sigma_k(\kappa|i)$ the k -th elementary symmetric polynomial in the variables $\kappa_1, \dots, \kappa_{i-1}, \kappa_{i+1}, \dots, \kappa_n$ and $\sigma_k(\kappa|ij)$ accordingly.

Lemma 13. *For $\lambda = 1, \dots, n-1$ suppose there exists some $\delta \leq 1$ such that $\kappa_\lambda/\kappa_1 \geq \delta$. There exists a sufficiently small positive constant δ' depending on δ, ϵ and the bounds for \mathcal{K} , such that if $\kappa_{\lambda+1}/\kappa_1 \leq \delta'$, we have*

$$A_i + B_i + C_i + D_i - E_i \geq 0 \quad \text{for } i = 1, \dots, \lambda.$$

Proof. This corresponds to [32, Lemma 4.3]. We highlight the main estimates in this proof. First of all, from [32, Equ. (4.16), (4.17)] one can extract the following estimate:

$$(5.2) \quad \begin{aligned} \|W\|^4 (B_i + C_i + D_i - E_i) &\geq \|W\|^2 \sum_{j \neq i} (\sigma_{n-1}(\kappa|j) - 2\sigma_{n-1}(\kappa|ij)) w_{jj;i}^2 \\ &\quad - w_{ii}^2 \sigma_n^{ii} w_{ii;i}^2 \\ &= \|W\|^2 \sum_{j \neq i} \sigma_{n-1}(\kappa|j) w_{jj;i}^2 - w_{ii}^2 \sigma_n^{ii} w_{ii;i}^2, \end{aligned}$$

since $\sigma_{n-1}(\kappa|ij) = 0$.

Now we show the right hand side of (5.2) is dominated by $\|W\|^4 A_i$. From (5.1) we get for all $1 \leq \lambda < n$ and for all $1 \leq i \leq n$:

$$\begin{aligned}
(5.3) \quad A_i &= \frac{(2-\varepsilon)w_{ii}}{\|W\|^2\sigma_n}((\sigma_n)_{;i})^2 - \frac{w_{ii}}{\|W\|^2} \sum_{p,q} \sigma_n^{pp,qq} w_{pp;i} w_{qq;i} \\
&\geq \frac{w_{ii}}{\|W\|^2} \left(1 + \frac{1-\varepsilon}{(n-1)(2-\varepsilon)}\right) \frac{\sigma_n((\sigma_\lambda)_{;i})^2}{\sigma_\lambda^2} \\
&\quad - \frac{w_{ii}}{\|W\|^2} \frac{\sigma_n \sum_{p,q} \sigma_\lambda^{pp,qq} w_{pp;i} w_{qq;i}}{\sigma_\lambda} \\
&= \frac{w_{ii}\sigma_n}{\|W\|^2\sigma_\lambda^2} \left[\left(1 + \frac{1-\varepsilon}{(n-1)(2-\varepsilon)}\right) \sum_a (\sigma_\lambda^{aa} w_{aa;i})^2 \right. \\
&\quad \left. + \frac{1-\varepsilon}{(n-1)(2-\varepsilon)} \sum_{a \neq b} \sigma_\lambda^{aa} \sigma_\lambda^{bb} w_{aa;i} w_{bb;i} \right. \\
&\quad \left. + \sum_{a \neq b} \left(\sigma_\lambda^{aa} \sigma_\lambda^{bb} - \sigma_\lambda \sigma_\lambda^{aa,bb} \right) w_{aa;i} w_{bb;i} \right].
\end{aligned}$$

For sufficiently small δ' and $\lambda = 1$ the simple estimates [32, Equ. (4.19), (4.20)] give

$$(5.4) \quad \|W\|^4 A_i \geq w_{ii}^2 \sigma_n^{ii} w_{11;i}^2 - C_\epsilon w_{ii} \sum_{a \neq 1} w_{aa;i}^2.$$

Combining this with (5.2) for $i = 1$ yields,

$$\begin{aligned}
(5.5) \quad \|W\|^2(A_1 + B_1 + C_1 + D_1 - E_1) &\geq \sum_{j \neq 1} \sigma_{n-1}(\kappa|j) w_{jj;1}^2 - \frac{C_\epsilon}{w_{11}} \sum_{j \neq 1} w_{jj;1}^2 \\
&= \sum_{j \neq 1} \left(\frac{\sigma_n}{w_{jj}} - \frac{C_\epsilon}{w_{11}} \right) w_{jj;1}^2 \\
&\geq \sum_{j \neq 1} \left(\frac{\sigma_n}{\delta' w_{11}} - \frac{C_\epsilon}{w_{11}} \right) w_{jj;1}^2,
\end{aligned}$$

which is non-negative for δ' sufficiently small. Hence the lemma is true in the case $\lambda = 1$.

For $\lambda > 1$ the series of elementary estimates [32, Equ. (4.22)-(4.27)] gives

$$\|W\|^4 A_i \geq w_{ii}^2 \sigma_n^{ii} \sum_{a \leq \lambda} w_{aa;i}^2 - \frac{w_{ii} C_\epsilon}{\delta^2} \sum_{a > \lambda} w_{aa;i}^2,$$

after having adapted ϵ if necessary and having chosen δ' sufficiently small again. Combining this last inequality with (5.2) for $1 \leq i \leq \lambda$ yields

$$\begin{aligned}
(5.6) \quad \|W\|^2(A_i + B_i + C_i + D_i - E_i) &\geq \sum_{j \neq i} \sigma_{n-1}(\kappa|j) w_{jj;i}^2 - \frac{C_\epsilon}{w_{ii} \delta^2} \sum_{j > \lambda} w_{jj;i}^2 \\
&\geq \sum_{j > \lambda} \left(\sigma_{n-1}(\kappa|j) - \frac{C_\epsilon}{w_{ii} \delta^2} \right) w_{jj;i}^2 \\
&\geq \sum_{j > \lambda} \left(\frac{\sigma_n}{w_{11} \delta'} - \frac{C_\epsilon}{w_{ii} \delta^2} \right) w_{jj;i}^2,
\end{aligned}$$

which is non-negative for small δ' for the same reason as in (5.5). This completes the proof. \square

Corollary 14. *There exist positive numbers $\delta_2, \dots, \delta_n$, depending only on the dimension, on ϵ and on the bounds for the Gauss curvature, such that either*

$$(5.7) \quad \kappa_i > \delta_i \kappa_1 \quad \forall 2 \leq i \leq n$$

or

$$(5.8) \quad A_i + B_i + C_i + D_i - E_i \geq 0 \quad \forall 1 \leq i \leq n.$$

Proof. Choosing $\lambda = 1$ and $\delta = 1$ in Lemma 13 yields the existence of δ' with the following property: if $\kappa_2/\kappa_1 \leq \delta'$, then

$$A_1 + B_1 + C_1 + D_1 - E_1 \geq 0.$$

Note that $\kappa_i \leq \kappa_2$ for $i \geq 2$. Choose $\delta_2 = \min\{\delta', 1/\sqrt{3}\}$. Therefore, in view of Lemma 12, $\kappa_2/\kappa_1 \leq \delta_2$ implies that

$$A_i + B_i + C_i + D_i - E_i \geq 0 \quad \forall i \geq 2.$$

We now apply induction, assuming we have constructed $\delta_2, \dots, \delta_j$. We may assume $\kappa_i > \delta_i \kappa_1$ for $2 \leq i \leq j$ otherwise $A_i + B_i + C_i + D_i - E_i \geq 0$ is already true for $2 \leq i \leq n$. Choose $\delta = \delta_j$ and $\lambda = j$ in Lemma 13 to get a δ' so that if $\kappa_{j+1} \leq \delta' \kappa_1$, then $A_i + B_i + C_i + D_i - E_i \geq 0$ holds for $1 \leq i \leq j$. Now in view of Lemma 12, taking $\delta_{j+1} = \min\{\delta', 1/\sqrt{3}\}$ gives $A_i + B_i + C_i + D_i - E_i \geq 0$ for $j \leq i \leq n$. \square

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